United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 28 January 2010

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (8-12 April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's IMO (to be held in Astana, Kazakhstan 6-12 July) will then be chosen.

Do not turn over until **told to do so.**

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2009/10 British Mathematical Olympiad
Round 2

1. There are 2010^{2010} children at a mathematics camp. Each has at most three friends at the camp, and if A is friends with B, then B is friends with A. The camp leader would like to line the children up so that there are at most 2010 children between any pair of friends. Is it always possible to do this?

2. In triangle ABC the centroid is G and D is the midpoint of CA. The line through G parallel to BC meets AB at E. Prove that $\angle AEC = \angle DGC$ if, and only if, $\angle ACB = 90°$. *The centroid of a triangle is the intersection of the three medians, the lines which join each vertex to the midpoint of the opposite side.*

3. The integer x is at least 3 and $n = x^6 - 1$. Let p be a prime and k be a positive integer such that p^k is a factor of n. Show that $p^{3k} < 8n$.

4. Prove that, for all positive real numbers x, y and z,
$$4(x+y+z)^3 > 27(x^2y + y^2z + z^2x).$$